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A THEORY ON CONSTRUCTING 2^{m-p} DESIGNS WITH GENERAL MINIMUM LOWER-ORDER CONFOUNDING*

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Choosing optimal designs under model uncertainty is an important step before conducting real experiments. The criterion of general minimum lower-order confounding (GMC) proposed by Zhang et al. (2008) controls the lower order factorial effects to be most slightly aliased with one another in an elaborate way, for choosing design. The construction of GMC 2^{m-p} designs with $n/2 \le m < n$, for any run-number $n = 2^{m-p}$, was considered with an approach of complementary design before. In this paper, we develop a theory of constructing GMC 2^{m-p} designs for $5n/16+1 \le m \le n/2$ as well as $m \ge n/2$. The results indicate that when $m \ge 5n/16+1$, every GMC design, up to isomorphism, simply consists of the last m columns of the saturated $2^{(n-1)-(n-1-m+p)}$ design with Yates order. Moreover, we prove that, at least for the following parameter intervals, every GMC design differs from minimum abberation design: $5n/16+1 \le m \le n/2-4$, and when $m \ge n/2$, $4 \le m + 2^r - n \le 2^{r-1} - 4$ with $r \ge 4$.

1. Introduction. Regular two-level fractional factorial designs are most commonly used in practical experiments. In the passed three decades, many statisticians payed a great of attention on selecting this kind of optimal designs, see Wu and Hammada (2000) and Mukerjee and Wu (2006) for a detailed review. Minimum aberation (MA) criterion is one of the most common criteria for this purpose. A large number of related papers appeared on this aspect since the landmark work Fries and Hunter (1980), such as Franklin (1984), Chen and Wu (1991), Chen et al. (1993), Chen and Hedayat (1996), Tang and Wu (1996), Zhang and Shao (2001), Butler (2003), Cheng and Tang (2005), Chen and Cheng (2006) and Xu and Cheng (2008).

However, MA criterion sometimes does not result in satisfactory designs. Wu and Chen (1992) introduced a notion of clear effect and noted that, MA criterion can not always find out the designs that possess maximum number of clear two-factor interactions (2fi's). Later on, more and more examples of design with the maximum numbers of clear main effects and 2fi's but different from MA design are found, see Wu and Hamada (2000) and Li et al. (2006). For recent developments in this area,

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we can refer to Chen and Hedayat (1998), Tang et al. (2002), Wu and Wu (2002), Ai and Zhang (2004), Chen et al. (2006), Yang et al. (2006) and Zhao and Zhang (2008). One usually calls the design with maximum numbers of clear main effects and 2fi's an optimal design under clear effects criterion.

The third one for selecting optimal designs is maximum estimation capacity criterion, firstly introduced by Sun (1993). Its aim is to estimate as many as possible models involving all the main effects and some 2fi's. For details we refer to Cheng and Mukerjee (1998) and Cheng et al. (1999).

Recently, by introducing a new pattern, called aliased effect-number pattern, Zhang et al. (2008) discussed advantages and disadvantages of the above criteria and proposed a new criterion, a general minimum lower-order confounding (GMC, for short) criterion. They have proved that, under the effect hierarchy principle the GMC criterion has much better performance than MA and clear effects criteria at finding optimal regular designs. Later on, Zhang and Mukerjee (2008) gave a further characterization to the GMC criterion via complementary set. The theory developed in their paper has been proved to be powerful when the number of factors in the complementary design is less than or equal to 15.

The purpose of the present paper is to contribute a theory on constructing GMC 2^{m-p} designs, which ideally works when the number of factors m is larger than $5n/16$, where n is the run-number of design.

The rest of this paper is organized as follows. In the next section, we review the definitions of MA and GMC criteria and introduce some notations. Especially, an important theorem is given in this section. In Section 3, a theory on constructing GMC designs, for $5n/16+1 \leq m \leq n/2$ as well as $m \geq n/2$, is developed. Some results on parameter intervals in which the GMC and MA designs are different are obtained in Section 4. In Appendix, we give a proof of the important theorem stated in Section 2.

2. Definitions, notations and an important theorem. Let D denote a 2^{m-p} design with m factors, $n = 2^{m-p}$ runs, and p independent defining words. The p independent defining words generate a group, called defining contrast group of D. We denote the factors by $1, 2, \ldots, m$ and also call $1, 2, \ldots, m$ letters. Every element not I (the identity element) in the group is called a word. The number of letters in a word is called its wordlength. Let $A_i(D)$ denote the number of words with length i in the defining contrast group of D. The vector $A(D) = (A_1(D), A_2(D), \ldots, A_m(D))$ is called wordlength pattern of D. The resolution of a design is the smallest r satisfying $A_r > 0$. A 2^{m-p} design with resolution r is denoted by 2_r^{m-p}. MA criterion is the rule to find design D, such that $(A_1(D), A_2(D), \ldots, A_m(D))$ is sequentially minimized in all possible regular designs with the same parameters.

We now review some concepts of the GMC criterion for two-level regular designs in Zhang et al. (2008). If an ith-order effect is aliased with k jth-order effects simultaneously, we say that the severe degree of the ith-order effect being aliased with jth-order effects is k. Let ${}^{\#}_{i}C_j^{(k)}$ denote the number of ith-order effects aliased with jth-order effects at degree k, and put

$$ {}^{\#}_{i}C_j = ({}^{\#}_{i}C_j^{(0)}, {}^{\#}_{i}C_j^{(1)}, \ldots, {}^{\#}_{i}C_j^{(K_j)}), $$

where $K_j = \binom{n}{j}$. The sequence or the set

$$ {}^{\#}C = ({}^{\#}_{1}C_1, {}^{\#}_{0}C_2, {}^{\#}_{1}C_2, {}^{\#}_{2}C_1, {}^{\#}_{2}C_2, {}^{\#}_{0}C_3, {}^{\#}_{1}C_3, {}^{\#}_{2}C_3, {}^{\#}_{3}C_1, {}^{\#}_{3}C_2, {}^{\#}_{3}C_3, \ldots) \quad (2.1) $$

is called an *aliased effect-number pattern* (AENP). In (2.1) as a sequence, the general rule of ${}^{\#}_{i}C_j$ being placed ahead of ${}^{\#}_{s}C_t$ is as follows: if $\max(i,j) < \max(s,t)$, or if $\max(i,j) = \max(s,t)$ and $i < s$, or if $\max(i,j) = \max(s,t)$, $i = s$ and $j < t$.

Zhang and Mukerjee (2008) found that some terms in (2.1) are uniquely determined by the terms before them. For example, ${}^{\#}_{j}C_1^{(1)} = \sum_{k \geq 1} k\, {}^{\#}_{1}C_j^{(k)}$. They further refined the sequence (2.1) to the simpler version

$$ {}^{\#}C = ({}^{\#}_{1}C_2, {}^{\#}_{2}C_2, {}^{\#}_{1}C_3, {}^{\#}_{2}C_3, {}^{\#}_{3}C_2, {}^{\#}_{3}C_3, \ldots). \quad (2.2) $$

The GMC criterion based on (2.2) is defined as follows.

DEFINITION 1. *Let ${}^{\#}C_l$ be the l-th component of ${}^{\#}C$, and ${}^{\#}C(D_1)$ and ${}^{\#}C(D_2)$ be the AENPs of designs D_1 and D_2, respectively. Suppose that ${}^{\#}C_t$ is the first component such that ${}^{\#}C_t(D_1)$ and ${}^{\#}C_t(D_2)$ are different. If ${}^{\#}C_t(D_1) > {}^{\#}C_t(D_2)$, then D_1 is said to have less general lower-order confounding than D_2. A design D is said to have general minimum lower-order confounding if no other design has less general lower-order confounding than D and such a design is called a GMC design.*

For convenience of presentation, we introduce some notations as follows. For a 2^{m-p} design, denote $q = m - p$ and let $1, \ldots, q$ stand for q independent factors. Further let H_r be the set containing all main effects $1, \ldots, r$ and all interactions between $1, \ldots, r$, $S_{qr} = H_q \backslash H_r$, $F_{qr} = \{q, qH_{r-1}\}$ and $T_r = \{r, rH_{r-1}\}$, where $qH_{r-1} = \{qd : d \in H_{r-1}\}$ and rH_{r-1} is similarly defined with conventions $F_{q1} = \{q\}$, $T_1 = \{1\}$, $qH_{1-1} = \{q\}$, and $1H_{1-1} = \{1\}$. Obviously, the designs F_{qr} and T_r with $r \geq 3$ are the saturated resolution IV design with r independent factors, which is unique up to isomorphism. We introduce both notations F_{qr} and T_r for easy presentation in Sections 3 and 4. Without loss of generality, suppose the columns in H_r, $r = 1, \ldots, q$ are written in Yates order. That is,

$$ H_1 = \{1\} \text{ and } H_r = \{H_{r-1}, r, rH_{r-1}\} \text{ for } r = 2, \ldots, q. $$

Throughout the paper, let S denote a design, a subset of H_q, with s factors (columns). All the results presented in this section are based on such S. Through the paper, we will treat the design S in which the s factors are independent as one with resolution at least IV, including $s \leq 3$, since it possesses the essential property of resolution at least IV: all main effects are not aliased with any other main effects and 2fi's.

For a given design $S \subset H_q$ and a $\gamma \in H_q$, define

$$B_i(S, \gamma) = \#\{(d_1, d_2, \ldots, d_i) : d_1, d_2, \ldots, d_i \in S, d_1 d_2 \cdots d_i = \gamma\},$$

where # denotes the cardinality of a set and $d_1 d_2 \cdots d_i$ means the ith order interaction of $d_1, d_2, \ldots, d_i$. By this definition, $B_i(S, \gamma)$ is the number of ith order interactions of S appearing in the alias set that contains γ. With the consideration above, the complementary set of a design is also a design. For the convenience of presentation in this paper, we define

$$\bar{g}(S) = \#\{\gamma : \gamma \in H_q \backslash S, B_2(S, \gamma) > 0\}. \tag{2.3}$$

Note that for the $g(S)$ defined in Zhang and Mukerjee (2008), just $g(S) = \bar{g}(H_q \backslash S)$ here. Minimizing $g(S)$ has been proved to be important when finding GMC designs, see Zhang and Mukerjee (2008). It is also a necessary condition in our theory.

In the following, we first give an important theorem, which studies the structure of a design S when $\bar{g}(S)$ is minimized, and will play a key role in developing the later theory on constructing GMC designs. The proof of the theorem is deferred to Appendix.

THEOREM 1. *Let $S \subset H_q$ be a design with s factors (columns). Then, under isomorphism, we have*

(a) if $2^{r-1} \leq s \leq 2^r - 1$ for some $r \leq q$ and $\bar{g}(S)$ is minimized in all the designs with s factors, then the S exactly has r independent factors and $S \subset H_r$;

(b) if $2^{r-2} + 1 \leq s \leq 2^{r-1}$ for some $r \leq q$ and $\bar{g}(S)$ is minimized in all the designs with s factors and resolution at least IV, then also the S exactly has r independent factors and $S \subset F_{qr}$ (or T_r);

(c) if $2^{r-2} + 1 \leq s \leq 2^{r-1}$ for some $r \leq q$, then S sequentially maximizes the components of

$$\{-\bar{g}(S), {}_2^{\#}C_2(S)\} \tag{2.4}$$

in all the designs with s factors and resolution at least IV if and only if the S consists of the first (or last) s columns of F_{qr} (or T_r) with Yates order, i.e., the S is any one of the four isomorphic constitutions.

For simplicity of statements hereafter, we will use some phrases to imply their complete expressions. For example, when we say that a design "sequentially maximizes (or minimizes) the components of" some sequence, we will simply say "maximizes" (or "minimizes") some sequence; "$\bar{g}(S)$ is minimized in all the designs with s factors" will be simply said as "$\bar{g}(S)$ is minimized"; also, we will mostly omit "up to isomorphism", since in this paper the designs are considered to be same if they are isomorphic. The readers can know their meanings in their corresponding contexts.

3. Theory on constructing GMC 2^{m-p} designs. In this section, let D be a 2^{m-p} design. Since the theoretic deductions of constructing GMC 2^{m-p} designs for the two cases $5n/16+1 \le m < n/2$ and $m \ge n/2$ are different, in the following we use two subsections separately to discuss them.

3.1. *GMC 2^{m-p} designs with $5n/16+1 \le m \le n/2$.* According to Theorem 1 in Zhang et al. (2008), obviously, if design D has GMC and $m \le n/2$ then its resolution must be at least IV. Note that any 2^{m-p}_{IV} design D with $5n/16+1 \le m \le n/2$ satisfies $D \subset F_{qq}$, see Bruen et al. (1998) and Butler (2007). Clearly, the number of factors in $F_{qq} \backslash D$ is less than that of D.

To study the construction of GMC designs, let us investigate the relationships between the AENP of D and that of $F_{qq} \backslash D$ first. We have the following.

LEMMA 1. *Let $D \subset F_{qq}$ be a 2^{m-p} design. Then*

$$\text{(a)}\ B_2(D,\gamma) = \begin{cases} 0, & \text{if } \gamma \in F_{qq}, \\ B_2(F_{qq}\backslash D,\gamma) + m - n/4, & \text{if } \gamma \in H_{q-1}, \end{cases}$$

$$\text{(b)}\ {}^{\#}_{1}C_2^{(k)}(D) = \begin{cases} m, & \text{if } k = 0, \\ 0, & \text{if } k \ge 1, \end{cases}$$

$$\text{(c)}\ {}^{\#}_{2}C_2^{(k)}(D) = \begin{cases} 0, & \text{if } k < m - n/4 - 1, \\ -(k+1)\bar{g}(F_{qq}\backslash D) + (k+1)(n/2-1), & \text{if } k = m - n/4 - 1, \\ (k+1)/\{k+1-(m-n/4)\}\, {}^{\#}_{2}C_2^{(k-m+n/4)}(F_{qq}\backslash D), & \text{if } k \ge m - n/4. \end{cases}$$

PROOF. For (a). The first part of (a) is obvious due to the structure of F_{qq}. For any $\gamma \in H_{q-1}$, there are $n/4$ pairs factors in F_{qq} whose interactions are aliased with γ. Among them $B_2(D,\gamma)$ pairs come from D, $B_2(F_{qq}\backslash D,\gamma)$ pairs come from $F_{qq}\backslash D$; and for the remaining pairs, one factor is from D and another one is from $F_{qq}\backslash D$. Therefore

$$B_2(D,\gamma) + B_2(F_{qq}\backslash D,\gamma) + m - 2B_2(D,\gamma) = n/4,$$

which is just the second equality of (a).

For (b). The result is obvious due to the structure of F_{qq}.

For (c). From the definition of ${}^{\#}_{2}C_2^{(k)}(D)$ and the result of (a), we have

$$\begin{aligned} {}^{\#}_{2}C_2^{(k)}(D) &= (k+1)\#\{\gamma \in H_q, B_2(D,\gamma) = k+1\} \\ &= (k+1)\#\{\gamma \in H_{q-1}, B_2(F_{qq}\backslash D,\gamma) = k+1-(m-n/4)\}. \end{aligned}$$

Thus the first and third equalities in (c) follow directly from the above equation and the definition of ${}_2^{\#}C_2^{(k)}(F_{qq}\backslash D)$. As for the second one, when $k = m - n/4 - 1$, by (a) and $\#\{H_{q-1}\} = n/2 - 1$ we have

$$\begin{aligned} {}_2^{\#}C_2^{(k)}(D) &= (k+1)\#\{\gamma \in H_{q-1}, B_2(F_{qq}\backslash D, \gamma) = 0\} \\ &= (k+1)(n/2-1) - (k+1)\#\{\gamma \in H_{q-1}, B_2(F_{qq}\backslash D, \gamma) > 0\} \\ &= (k+1)(n/2-1) - (k+1)\#\{\gamma \in H_q\backslash(F_{qq}\backslash D), B_2(F_{qq}\backslash D, \gamma) > 0\} \\ &= (k+1)(n/2-1) - (k+1)\bar{g}(F_{qq}\backslash D). \end{aligned}$$

Then the second equality in (c) follows. □

Obviously, the above lemma yields the following lemma, which can be easily used to construct GMC designs when the number of factors in $F_{qq}\backslash D$ is small.

LEMMA 2. *Suppose D is a 2^{m-p} design with $5n/16 + 1 \le m \le n/2$. The design D has GMC if $D \subset F_{qq}$ and it uniquely maximizes*

$$\{-\bar{g}(F_{qq}\backslash D), {}_2^{\#}C_2(F_{qq}\backslash D)\}. \tag{3.1}$$

Combining Lemma 2 and Part (c) of Theorem 1, we can get the following valuable result.

THEOREM 2. *Suppose the columns in H_q and F_{qq} are written in Yates order. For $5n/16 + 1 \le m \le n/2$, the GMC 2^{m-p} design is just the design that consists of the last m columns in H_q or F_{qq}.*

PROOF. Suppose $2^{r-2} + 1 \le n/2 - m \le 2^{r-1}$ for some r. Letting $S = F_{qq}\backslash D$, $s = n/2 - m$ and applying Part (c) of Theorem 1, the design $F_{qq}\backslash D$ consisting of the first $n/2 - m$ columns of F_{qr} will uniquely maximize the sequence (3.1). When H_q and F_{qq} are written in Yates order, the first $n/2 - m$ columns of F_{qr} are also the first $n/2 - m$ columns of F_{qq}. Hence the GMC design D consists of the last m columns of F_{qq}. Noting that the last m columns of F_{qq} are just the last m columns of H_q, then the result follows directly. □

To illustrate the construction method in Theorem 2, let us see the following example.

EXAMPLE 1. Suppose that we need to get a GMC design with $m = n/2 - 5$ factors, where n can be 32, 64, or 2048 whatever as long as $5n/16 + 1 \le m \le n/2$. Let us take a saturated resolution IV design F_{q4} with Yates order, which has 4 (4 is enough since $2^{4-1} \ge 5$) independent factors. The F_{q4} can be written as

$$F_{q4} = \{q, 1q, 2q, 12q, 3q, 13q, 23q, 123q\}.$$

According to Theorem 2, when $n/2 - m = 5$,

$$D = F_{qq} \backslash \{q, 1q, 2q, 12q, 3q\}$$

is just a GMC design. Especially, if $n = 32$, then $m = 11$ and the GMC design $D = \{135, 235, 1235, 45, 145, 245, 1245, 345, 1345, 2345, 12345\}$.

If $m = n/2 - 10$ and $n \geq 64$, we need to take a saturated resolution IV design with 5 independent factors F_{q5} with Yates order, which can be written as

$$\begin{aligned} F_{q5} \; = \; & \{q, 1q, 2q, 12q, 3q, 13q, 23q, 123q, \\ & 4q, 14q, 24q, 124q, 34q, 134q, 234q, 1234q\}. \end{aligned}$$

Then, according to Theorem 2, since $n/2 - m = 10$, we get that

$$D = F_{qq} \backslash \{q, 1q, 2q, 12q, 3q, 13q, 23q, 123q, 4q, 14q\},$$

is just a GMC design. □

3.2. *GMC 2^{m-p} designs with $m \geq n/2$.* When $m \geq n/2$, Zhang and Mukerjee (2008) found that if D has GMC then $\bar{g}(H_q \backslash D)$ is minimized. According to Part (a) of Theorem 1, when $2^{r-1} \leq n - 1 - m \leq 2^r - 1$ for some r, $H_q \backslash D$ has r independent factors. Therefore $H_q \backslash D \subset H_r$ and $S_{qr} \subset D$, where S_{qr} is defined in Section 2. When the number of factors in $D \backslash S_{qr}$ is small, it will be convenient for us to construct GMC designs based on $D \backslash S_{qr}$. Hence the relationship between the AENPs of D and $D \backslash S_{qr}$ will be very helpful. The next lemma first studies the connection between $B_2(D, \gamma)$ and $B_2(D \backslash S_{qr}, \gamma)$.

LEMMA 3. *Suppose D is a 2^{m-p} design with $S_{qr} \subset D$. Then*

(a) if $\gamma \in S_{qr}$, $B_2(D, \gamma) = m - n/2$;
(b) if $\gamma \in H_r$, $B_2(D, \gamma) = B_2(D \backslash S_{qr}, \gamma) + n/2 - 2^{r-1}$.

PROOF. For (a). From the structure of S_{qr}, we have for any $\gamma \in S_{qr}$,

$$\begin{aligned} B_2(D, \gamma) \; = \; & \#\{(d_1, d_2) : \gamma = d_1 d_2, \; d_1 \in D \backslash S_{qr}, \; d_2 \in S_{qr}\} \\ & + \#\{(d_1, d_2) : \gamma = d_1 d_2, \; d_1 \in S_{qr}, \; d_2 \in S_{qr}\}. \end{aligned}$$

For any $d_1 \in D \backslash S_{qr}$, we can uniquely determine $d_2 = d_1 \gamma$ in S_{qr}. So

$$\#\{(d_1, d_2) : \gamma = d_1 d_2, \; d_1 \in D \backslash S_{qr}, \; d_2 \in S_{qr}\} = m - (n - 2^r).$$

Note that for any $\gamma \in S_{qr}$, there are $n/2 - 1$ pairs factors in H_q whose interactions are aliased with γ. Among them, there are $2^r - 1$ pairs with one factor from H_r and another one from S_{qr}; for the remaining $n/2 - 2^r$ pairs, both factors are from S_{qr}. Therefore

$$\#\{(d_1, d_2) : \gamma = d_1 d_2, \; d_1 \in S_{qr}, \; d_2 \in S_{qr}\} = n/2 - 2^r$$

and

$$B_2(D,\gamma) = m - (n - 2^r) + n/2 - 2^r = m - n/2.$$

Then the result in (a) follows.

For (b). For any $\gamma \in H_r$, we have

$$\begin{aligned} B_2(D,\gamma) &= \#\{(d_1,d_2) : \gamma = d_1d_2,\ d_1 \in D\backslash S_{qr},\ d_2 \in D\backslash S_{qr}\} \\ &\quad +\#\{(d_1,d_2) : \gamma = d_1d_2,\ d_1 \in S_{qr},\ d_2 \in S_{qr}\} \\ &= B_2(D\backslash S_{qr},\gamma) + \#\{(d_1,d_2) : \gamma = d_1d_2,\ d_1 \in S_{qr},\ d_2 \in S_{qr}\}, \end{aligned}$$

where the second equality is from the definition of $B_2(D\backslash S_{qr},\gamma)$. Note that for any $\gamma \in H_r$, there are $n/2-1$ pairs factors in H_q whose interactions are aliased with γ. Among them, there are $(2^r-2)/2 = 2^{r-1}-1$ pairs from H_r and $n/2-2^{r-1}$ pairs from S_{qr}. Hence

$$\#\{(d_1,d_2) : \gamma = d_1d_2,\ d_1 \in S_{qr},\ d_2 \in S_{qr}\} = n/2 - 2^{r-1}$$

and

$$B_2(D,\gamma) = B_2(D\backslash S_{qr},\gamma) + n/2 - 2^{r-1}.$$

This finishes the proof of (b). □

The above lemma can be applied to yield the expressions of the leading terms of AENP of a design $D \supset S_{qr}$ for some r, shown in the lemma below, in terms of that of the design $D\backslash S_{qr}$.

LEMMA 4. *Suppose $D = \{S_{qr}, D\backslash S_{qr}\}$. Then*

(a) $$ {}^{\#}_{1}C_2^{(k)}(D) = \begin{cases} constant, & if\ k < n/2 - 2^{r-1}, \\ {}^{\#}_{1}C_2^{(k-n/2+2^{r-1})}(D\backslash S_{qr}) + constant, & if\ k \geq n/2 - 2^{r-1}, \end{cases}$$

(b) $$ {}^{\#}_{2}C_2^{(k)}(D) = \begin{cases} constant, & if\ k < n/2 - 2^{r-1} - 1, \\ -(k+1)\bar{g}(D\backslash S_{qr}) + (k+1){}^{\#}_{1}C_2^{(0)}(D\backslash S_{qr}) & \\ \ +constant, & if\ k = n/2 - 2^{r-1} - 1, \\ (k+1)/(k-n/2+2^{r-1}+1){}^{\#}_{2}C_2^{(k-n/2+2^{r-1})}(D\backslash S_{qr}) & \\ \ +constant, & if\ k \geq n/2 - 2^{r-1}, \end{cases}$$

where the constant's are non-negative values only depending on m, k and n.

PROOF. For (a). From the definition of ${}^{\#}_{1}C_2^{(k)}(D)$, we have

$$\begin{aligned} {}^{\#}_{1}C_2^{(k)}(D) &= \#\{\gamma : \gamma \in S_{qr}, B_2(D,\gamma) = k\} \\ &\quad +\#\{\gamma : \gamma \in D\backslash S_{qr}, B_2(D,\gamma) = k\}. \end{aligned}$$

From Parts (a) and (b) of Lemma 3, we get that

$$\begin{aligned} {}^{\#}_{1}C_2^{(k)}(D) &= I(m - n/2 = k) \times (n - 2^r) \\ &\quad +\#\{\gamma : \gamma \in D\backslash S_{qr}, B_2(D\backslash S_{qr},\gamma) + n/2 - 2^{r-1} = k\}, \end{aligned}$$

where $I(\cdot)$ is the indicator function. Part (a) follows directly.

For (b). By the definition of ${}_2^{\#}C_2^{(k)}(D)$, we have

$$\begin{aligned}{}_2^{\#}C_2^{(k)}(D) &= (k+1)\#\{\gamma:\gamma\in S_{qr}, B_2(D,\gamma)=k+1\}\\ &\quad +(k+1)\#\{\gamma:\gamma\in H_r, B_2(D,\gamma)=k+1\}.\end{aligned}$$

Using Parts (a) and (b) of Lemma 3, the above equation reduces to

$$\begin{aligned}{}_2^{\#}C_2^{(k)}(D) &= I(m-n/2=k+1)\times(k+1)(n-2^r)\\ &\quad +(k+1)\#\{\gamma:\gamma\in H_r, B_2(D\backslash S_{qr},\gamma)=k+1-n/2+2^{r-1}\}.\end{aligned}$$

The first and third equalities of (b) follow directly from the above equation and the definition of ${}_2^{\#}C_2^{(k)}(D\backslash S_{nr})$.

For the second equality of (b), when $k=n/2-2^{r-1}-1$, we have

$$\begin{aligned}{}_2^{\#}C_2^{(k)}(D) &= (k+1)\#\{\gamma:\gamma\in H_r, B_2(D\backslash S_{qr},\gamma)=0\}+constant\\ &= (k+1)\#\{\gamma:\gamma\in D\backslash S_{qr}, B_2(D\backslash S_{qr},\gamma)=0\}\\ &\quad +(k+1)\#\{\gamma:\gamma\in H_q\backslash D, B_2(D\backslash S_{qr},\gamma)=0\}+constant.\end{aligned}$$

Note that from the definitions of ${}_1^{\#}C_2^{(k)}(D\backslash S_{qr})$ and $\bar{g}(\cdot)$ in (2.3), we have

$$\#\{\gamma:\gamma\in D\backslash S_{qr}, B_2(D\backslash S_{qr},\gamma)=0\}={}_1^{\#}C_2^{(0)}(D\backslash S_{qr})$$

and

$$\begin{aligned}&\#\{\gamma:\gamma\in H_q\backslash D, B_2(D\backslash S_{qr},\gamma)=0\}\\ &\quad= (n-1-m)-\#\{\gamma:\gamma\in H_q\backslash D, B_2(D\backslash S_{qr},\gamma)>0\}\\ &\quad= (n-1-m)-\#\{\gamma:\gamma\in S_{qr}\cup(H_q\backslash D), B_2(D\backslash S_{qr},\gamma)>0\}\\ &\quad= (n-1-m)-\#\{\gamma:\gamma\in H_q\backslash(D\backslash S_{qr}), B_2(D\backslash S_{qr},\gamma)>0\}\\ &\quad= (n-1-m)-\bar{g}(D\backslash S_{qr}).\end{aligned}$$

The second equality above is from the structures of S_{qr} and $D\backslash S_{qr}$. Then the second equality of (b) follows directly. □

The following lemma immediately follows from the above lemma, which can be easily used to construct GMC designs when the number of factors in $D\backslash S_{qr}$ is small.

LEMMA 5. *Suppose D is a 2^{m-p} design with $2^{r-1}\le n-1-m\le 2^r-1$ for some $r\le q-1$. The design D has GMC if $S_{qr}\subset D$ and it is unique one that maximizes*

$$\{{}_1^{\#}C_2(D\backslash S_{qr}),\ -\bar{g}(D\backslash S_{qr}),\ {}_2^{\#}C_2(D\backslash S_{qr})\}. \tag{3.2}$$

When $2^{r-1} \leq n-1-m \leq 2^r-1$, there are r independent factors in H_r and $m+2^r-n$ ($< 2^{r-1}$) factors in $D \backslash S_{nr}$. So we can find a design with resolution at least IV and $m+2^r-n$ factors in H_r. Note that ${}_1^{\#}C_2(D\backslash S_{qr})$ is maximized if $D\backslash S_{nr}$ has resolution at least IV. The next two terms after ${}_1^{\#}C_2(D\backslash S_{qr})$ are $-\bar{g}(D\backslash S_{qr})$ and ${}_2^{\#}C_2(D\backslash S_{qr}\}$. Applying Part (c) of Theorem 1, we get a result similar to Theorem 2.

THEOREM 3. *Suppose the columns in H_q are written in Yates order. For $m \geq n/2$, the GMC 2^{m-p} design is just the design that consists of the last m columns in H_q.*

PROOF. Suppose $2^{r-1} \leq n-1-m \leq 2^r-1$ for some $r \leq q-1$ and $S_{qr} \subset D$. Let $f_r = m+2^r-n$, which is the number of columns in $D\backslash S_{qr}$. Then $0 \leq f_r \leq 2^{r-1}-1$.

When $f_r = 0$ or 1, then $D = S_{qr}$ or $S_{qr} \cup \{12\cdots r\}$, the result is obvious. Next consider $2^{l-2}+1 \leq f_r \leq 2^{l-1}$ for some $2 \leq l \leq r$. Letting $S = D\backslash S_{qr}$, $s = f_r$ and applying Part (c) of Theorem 1, we have that if $D\backslash S_{qr}$ consists of the first f_r columns of T_l, then $D\backslash S_{qr}$ uniquely maximizes the sequence (3.2). Here T_l is defined in Section 2. When H_q is written in Yates order, the design consisting of the first f_r columns of T_l is isomorphic to the one consisting of the first f_r columns of T_r. Note that Theorem 1 implies that the design consisting of the first f_r columns of T_r is isomorphic to the one consisting of the last f_r columns of T_r (see Part (c) of the theorem). Therefore, if $D\backslash S_{qr}$ consists of the last f_r columns of T_r, then $D\backslash S_{qr}$ uniquely maximizes the sequence (3.2) under isomorphism. Combining with S_{qr}, the design consisting of the last m columns of H_q has GMC. □

Next let us use an example to illustrate the construction method in Theorem 3.

EXAMPLE 2. Consider the case when $n-1-m = 2^{r-1}+3$ with $r \geq 3$. According to Theorem 3, by deleting the first $2^{r-1}+3$ columns from H_q, the resulted design

$$D = H_q \backslash (H_{r-1} \cup \{r, 1r, 2r, 12r\})$$

is just a GMC design. □

4. When a GMC design will be different from MA design. In this section we examine that under what parameters the GMC and MA designs are certainly different. Our results are shown in Theorems 4 and 5 below.

THEOREM 4. *Suppose $D \subset F_{qq}$ is a 2_{IV}^{m-p} design with $5n/16+1 \leq m \leq n/2$. When $m \leq n/2-4$, an MA design must not maximize ${}_2^{\#}C_2(D)$ and hence any GMC design differs from MA design.*

PROOF. When $5n/16+1 \leq m \leq n/2$, Butler (2003) proved that if D is an MA design, then $D \subset F_{qq}$ and $F_{qq}\backslash D$ has MA among designs in F_{qq}. So the number of independent factors in $F_{qq}\backslash D$ is $\min(n/2-m, q)$.

According to Part (c) of Lemma 1, if a design $D \subset F_{qq}$ and maximizes ${}_2^{\#}C_2(D)$, then $\bar{g}(F_{qq}\backslash D)$ is minimized. Due to the structure of F_{qq}, the design $F_{qq}\backslash D$ has resolution at least IV. Applying Part (b) of Theorem 1, we get that the maximum number of independent factors in $F_{qq}\backslash D$ is at most $\lfloor \log_2(n/2-m-1)\rfloor+2$, where $\lfloor x \rfloor$ denotes the largest integer which is smaller or equal to x. When $5n/16+1 \le m \le n/2$,

$$\lfloor \log_2(n/2-m-1)\rfloor+2 \le \lfloor \log_2(n/2-5n/16-2)\rfloor+2 = \lfloor \log_2(3n/16-2)\rfloor+2 < q.$$

Also when $m \le n/2-4$, $\lfloor \log_2(n/2-m-1)\rfloor+2 < n/2-m$. Therefore, the number of independent factors in $F_{qq}\backslash D$ is less than $\min(n/2-m, q)$ if D maximizes ${}_2^{\#}C_2(D)$. Thus, the theorem follows from the above argument. □

Similar to Theorem 4, we have the following theorem, which tells us that, when $m \ge n/2$, on what parameter intervals, the GMC and MA designs are different.

THEOREM 5. *Suppose* $2^{r-1} \le n-1-m \le 2^r-1$ *for some* r. *When* $4 \le m+2^r-n \le 2^{r-1}-4$ *with* $4 \le r \le q-1$, *any GMC design differs from MA design.*

PROOF. When $m \ge n/2$, using Lemma 4 of Chen and Hedayat (1996), Butler (2003) proved that if D is an MA design, then $F_{qq} \subset D$ and $D\backslash F_{qq}$ has MA among the designs in H_{q-1}. Repeatedly applying this result and Lemma 4 of Chen and Hedayat (1996), one can prove a stronger result that if D is an MA design, then $S_{qr} \subset D$ and $D\backslash S_{qr}$ has MA among the designs in H_r. So the number of independent factors in $D\backslash S_{qr}$ is $\min(m+2^r-n, r)$ if D has MA.

According to Lemma 5 and the discussion afore Theorem 3, if D has GMC, then $S_{qr} \subset D$, $D\backslash S_{qr}$ has resolution at least IV and $\bar{g}(D\backslash S_{nr})$ is minimized. By Part (b) of Theorem 1, the number of independent factors in $D\backslash S_{qr}$ is at most $\lfloor \log_2(m+2^r-n-1)\rfloor+2$. When $4 \le m+2^r-n \le 2^{r-2}$ with $r \ge 4$, we can easily check that

$$\lfloor \log_2(m+2^r-n-1)\rfloor+2 < \min(m+2^r-n, r)$$

and hence in this region every GMC design differs from MA design. When $2^{r-2}+1 \le m+2^r-n \le 2^{r-1}-4$ with $r \ge 5$, there are r independent factors in $D\backslash S_{qr}$ and $D\backslash S_{qr} \subset T_r$, see Part (b) of Theorem 1. By Lemma 1 (a) with q being taken as r and the condition $2^{r-2}+1 \le m+2^r-n \le 2^{r-1}-4$, for any $\gamma \in H_{r-1}$,

$$B_2(D\backslash S_{qr}, \gamma) = B_2(T_r\backslash(D\backslash S_{qr}), \gamma) + m + 2^r - n - 2^{r-2} \ge 1$$

and therefore $\bar{g}(D\backslash S_{qr}) = 2^{r-1}-1$, which is a constant. So if D has GMC, then $D\backslash S_{qr} \subset T_r$ and it maximizes ${}_2^{\#}C_2(D\backslash S_{qr})$. Similarly to Theorem 4, we can prove that $D\backslash S_{qr}$ is not an MA design among designs in H_r and hence D differs from MA design. □

Zhang and Mukerjee (2008) found that when $n-1-m=11$, the GMC and MA designs are different, which is the special case of Theorem 5 at $r=4$ to the moment.

APPENDIX A: PROOFS OF THEOREM 1.

The global line of proving the theorem is that, firstly we prove Part (a) and then use the result of Part (a) to prove Part (b), finally use the result of Part (b) to prove Part (c).

Recall the notations H_r, F_{qr} and T_r given in Section 2. For convenience of presentation below, we introduce the notation $Q_1 \times Q_2 = \{d_1 d_2 : d_1 \in Q_1,\ d_2 \in Q_2\}$, where $Q_1, Q_2 \subset H_q$. Particularly, denote $dQ = \{d\} \times Q$ for $d \in H_q$ and $Q \subset H_q$.

Proof for Part (a) of Theorem 1

Note that, when $r = q$, Part (a) of the theorem is obviously valid, since any design S with s factors from H_q, satisfying $2^{q-1} \le s \le 2^q - 1$, has exactly q independent factors and $S \subset H_q$. So, we only need to consider the case $r \le q - 1$.

We will use apagogical approach to prove the case. To carry out this point, in the following we first prove some general results.

Suppose that $S_1 \subset H_q$ is a design with s factors, where $2^{r-1} \le s \le 2^r - 1$ for some $r \le q-1$, and has $h+1$ $(r \le h \le q-1)$ independent factors. Let a denote the factor q. Under isomorphism, we can assume $a \in S_1$ and S_1 can be represented as

$$S_1 = Q \cup \{a, ab_1, ab_2, \ldots, ab_l\}, \tag{A.1}$$

where Q is a subset of H_h and has h independent factors, and $\{b_1, \ldots, b_l\} \subset H_h$. Without loss of generality, we assume that $\{b_1, \ldots, b_t\} \subset Q$ and $\{b_{t+1}, \ldots, b_l\} \subset H_h \backslash Q$, and consider another set

$$S_2 = Q \cup \{a, ab_1, \ldots, ab_t\} \cup \{b_{t+1}, \ldots, b_l\}. \tag{A.2}$$

We have the following lemma.

LEMMA 6. *Suppose that S_1 and S_2 are defined in (A.1) and (A.2) respectively, then $\bar{g}(S_2) \le \bar{g}(S_1)$.*

PROOF. Denote $Q_1 = \{a, ab_1, ab_2, \ldots, ab_t\}$ and $Q_2 = \{ab_{t+1}, \ldots, ab_l\}$. Then we have $S_1 = Q \cup Q_1 \cup Q_2$ and $S_2 = Q \cup Q_1 \cup aQ_2$. Let $P = H_q \backslash (S_1 \cup S_2)$, where $S_1 \cup S_2 = Q \cup Q_1 \cup Q_2 \cup aQ_2$ in which the four sets are mutually exclusive. According to the definitions of $\bar{g}(S_1)$ and $\bar{g}(S_2)$, we easily get

$$\begin{aligned} \bar{g}(S_1) = &\ \#\{\gamma : \gamma \in P, B_2(S_1, \gamma) > 0\} \\ &+ \#\{\gamma : \gamma \in aQ_2, B_2(S_1, \gamma) > 0\} \triangleq g_{11} + g_{12} \end{aligned}$$

and

$$\begin{aligned} \bar{g}(S_2) = &\ \#\{\gamma : \gamma \in P, B_2(S_2, \gamma) > 0\} \\ &+ \#\{\gamma : \gamma \in Q_2, B_2(S_2, \gamma) > 0\} \triangleq g_{21} + g_{22}. \end{aligned}$$

For any $\gamma \in Q_2$, $\gamma = a(a\gamma)$, where $a \in Q_1 \subset S_2$ and $a\gamma \in aQ_2 \subset S_2$, we have $B_2(S_2,\gamma) > 0$. From the definition of g_{22}, we get $g_{22} = \#\{Q_2\}$. Similarly, $g_{12} = \#\{aQ_2\}$ follows. It is easy to check that $\#\{Q_2\} = \#\{aQ_2\}$, which leads to $g_{12} = g_{22}$.

Now we pick-up the three sets:

$$P_1 = \{\gamma : \gamma \in P, B_2(S_1,\gamma) > 0 \text{ and } B_2(S_2,\gamma) > 0\},$$
$$P_2 = \{\gamma : \gamma \in P, B_2(S_1,\gamma) > 0 \text{ and } B_2(S_2,\gamma) = 0\}, \text{ and}$$
$$P_3 = \{\gamma : \gamma \in P, B_2(S_1,\gamma) = 0 \text{ and } B_2(S_2,\gamma) > 0\}.$$

Clearly, P_1, P_2 and P_3 are mutually exclusive and we have $g_{11} = \#\{P_1\} + \#\{P_2\}$ and $g_{21} = \#\{P_1\} + \#\{P_3\}$. So, to finish the proof, it suffices to show the result $\#\{P_2\} \geq \#\{P_3\}$ or a stronger result: if $\gamma \in P_3$ then $a\gamma \in P_2$.

To do this, we note that, if $\gamma \in P_3$, then γ must not be an interaction of any two factors in $Q \cup Q_1 \cup Q_2$ but an interaction of some two factors in $Q \cup Q_1 \cup aQ_2$. Therefore, γ must be an interaction of two factors with one coming from aQ_2 and the other coming from Q or Q_1. If $\gamma \in aQ_2 \times Q_1 = Q_2 \times aQ_1 \subset Q_2 \cup (Q_2 \times Q)$ or $\gamma \in aQ_2 \times \{b_1,\ldots,b_t\} = Q_2 \times \{ab_1,\ldots,ab_t\} \subset Q_2 \times Q_1$, where $\{b_1,\ldots,b_t\} \subset Q$, then $\gamma \notin P$ or $B_2(S_1,\gamma) > 0$, which contradicts the assumption $\gamma \in P_3$. So, it must be to have $\gamma \in aQ_2 \times (Q\backslash\{b_1,\ldots,b_t\}) \subset H_h$. Because of this, we have $a\gamma \in Q_2 \times (Q\backslash\{b_1,\ldots,b_t\})$, which implies $B_2(S_1,a\gamma) > 0$. The remainder is to prove that, for the $a\gamma$, we have $a\gamma \in P$ and $B_2(S_2,a\gamma) = 0$. For the former, it is easy to be validated. We only show $B_2(S_2,a\gamma) = 0$ below.

We use the reduction to absurdity to prove the point. Suppose $B_2(S_2,a\gamma) > 0$. Since $\gamma \in H_h$, we have $a\gamma \in aH_h$ and $a\gamma \in Q_1 \times (Q \cup aQ_2)$. Thus, there are only the following two possibilities: $a\gamma \in Q_1 \times Q$ or $a\gamma \in Q_1 \times aQ_2$. However, if $a\gamma \in Q_1 \times Q$, then $\gamma \in aQ_1 \times Q \subset Q \cup (Q \times Q)$, or if $a\gamma \in Q_1 \times aQ_2$, then $\gamma \in Q_1 \times Q_2$. Any one of the two cases implies that $\gamma \notin P$ or $B_2(S_1,\gamma) > 0$, which contradicts the assumption $\gamma \in P_3$. Lemma 6 is proved. □

Lemma 6 indicates that, if the design S_1 is transformed into the design S_2, i.e. the elements $ab_{t+1},\ldots,ab_l$ in S_1, which are out of H_h, are substituted by the elements $b_{t+1},\ldots,b_l$, which are in H_h, then $\bar{g}(S_2) \leq \bar{g}(S_1)$.

In the following study, we join Q and aQ_2 together and still denote it by Q. Without loss of generality, we assume that S_2 has the form

$$S_2 = Q \cup \{a, ab_1, \ldots, ab_t\}, \tag{A.3}$$

where $Q \subset H_h$ and has h independent factors, and $\{b_1,\ldots,b_t\} \subset Q$. When $2^{r-1} \leq s \leq 2^r - 1$, the number of factors in Q is smaller than $2^r - 1$. Therefore there are at least two factors c_1 and c_2 in Q such that $c = c_1c_2 \notin Q$. Under isomorphism, we can assume that there is some t_0 such that

$$\{c, cb_1, cb_2, \ldots, cb_{t_0}\} \subset H_h\backslash Q \quad \text{and} \quad \{cb_{t_0+1},\ldots,cb_t\} \subset Q.$$

Denote

$$S_3 = Q \cup \{c, cb_1, cb_2, \ldots, cb_{t_0}\} \cup \{ab_{t_0+1}, \ldots, ab_t\}. \tag{A.4}$$

We have one more result as follows.

LEMMA 7. *Suppose that S_2 and S_3 are defined in (A.3) and (A.4) respectively. Then $\bar{g}(S_3) \leq \bar{g}(S_2)$. Especially, if $t_0 = t$ the strict inequality $\bar{g}(S_3) < \bar{g}(S_2)$ is valid.*

PROOF. Let $Q_1 = \{a, ab_1, ab_2, \ldots, ab_{t_0}\}$ and $Q_2 = \{ab_{t_0+1}, \ldots, ab_t\}$. Then $S_2 = Q \cup Q_1 \cup Q_2$ and $S_3 = Q \cup acQ_1 \cup Q_2$. Also denote $P = H_q \backslash (S_2 \cup S_3)$, where $S_2 \cup S_3 = Q \cup Q_1 \cup acQ_1 \cup Q_2$ in which the four parts are mutually exclusive. According to the definitions of $\bar{g}(S_2)$ and $\bar{g}(S_3)$, we have

$$\begin{aligned}\bar{g}(S_2) = \ & \#\{\gamma : \gamma \in P, B_2(S_2, \gamma) > 0\} \\ & + \#\{\gamma : \gamma \in acQ_1, B_2(S_2, \gamma) > 0\} \triangleq g_{21} + g_{22}\end{aligned}$$

and

$$\begin{aligned}\bar{g}(S_3) = \ & \#\{\gamma : \gamma \in P, B_2(S_3, \gamma) > 0\} \\ & + \#\{\gamma : \gamma \in Q_1, B_2(S_3, \gamma) > 0\} \triangleq g_{31} + g_{32}.\end{aligned}$$

Now let

$$\begin{aligned}&P_1 = \{\gamma : \gamma \in P, B_2(S_2, \gamma) > 0 \text{ and } B_2(S_3, \gamma) > 0\}, \\ &P_2 = \{\gamma : \gamma \in P, B_2(S_2, \gamma) > 0 \text{ and } B_2(S_3, \gamma) = 0\}, \text{ and} \\ &P_3 = \{\gamma : \gamma \in P, B_2(S_2, \gamma) = 0 \text{ and } B_2(S_3, \gamma) > 0\}.\end{aligned}$$

Then, P_1, P_2 and P_3 are mutually exclusive, $g_{21} = \#\{P_1\} + \#\{P_2\}$, and $g_{31} = \#\{P_1\} + \#\{P_3\}$.

If $t_0 = t$, then $Q_2 = \emptyset$, the empty set, and $S_3 \subset H_h$. As a result, for any $\gamma \in Q_1$, we have $B_2(S_3, \gamma) = 0$ and hence $g_{32} = 0$. On the other hand, because there is $c \in acQ_1$ such that $B_2(S_2, c) > 0$, it leads to $g_{22} \geq 1$. Thus, to prove $\bar{g}(S_3) < \bar{g}(S_2)$, it suffices to show that $\#\{P_2\} \geq \#\{P_3\}$ or a stronger result: if $\gamma \in P_3$ then $ac\gamma \in P_2$. By the same argument as in proving Lemma 6, it is easy to know that if $\gamma \in P_3$ then $\gamma \in acQ_1 \times Q \subset H_h$. From this, it directly follows that $ac\gamma \in Q_1 \times Q$, then it is easy to verify $B_2(S_2, ac\gamma) > 0$ and $ac\gamma \in P$. Note that, $B_2(S_3, ac\gamma) = 0$ is straightforward since $S_3 \subset H_h$ and $ac\gamma \notin H_h$. In this way, the second half result of Lemma 7 follows.

In the following let us consider the case $t_0 < t$. Actually the proof for this case is very similar to that for $t_0 = t$. It only needs one more condition $acQ_2 \subset Q$, however it is just a simple fact from the definition of S_3.

To make it clear, let us take two steps. Firstly, we show the fact: for any $\gamma \in Q_1$, if $B_2(S_3, \gamma) > 0$ then $B_2(S_2, ac\gamma) > 0$ and hence $g_{32} \leq g_{22}$.

Note that, if $\gamma \in Q_1$ and $B_2(S_3, \gamma) > 0$, then $\gamma \in Q_2 \times (Q \cup acQ_1)$. Based on this, the above fact immediately follows, since we have that, if $\gamma \in Q_2 \times Q$ then $ac\gamma \in acQ_2 \times Q \subset Q \times Q$, or if $\gamma \in Q_2 \times acQ_1$ then $ac\gamma \in Q_2 \times Q_1$, both lead to $B_2(S_2, ac\gamma) > 0$.

Next, let us show the fact: for any $\gamma \in P_3$, then $ac\gamma \in P_2$ and hence $g_{31} \le g_{21}$.

Again, note that for any $\gamma \in P_3$ we have $\gamma \in acQ_1 \times (Q \cup Q_2)$. From this, we can first conclude $\gamma \notin acQ_1 \times Q_2 = Q_1 \times acQ_2 \subset Q_1 \times Q$, it is because if not then $B_2(S_2, \gamma) > 0$, but under given $\gamma \in P_3$ it is impossible. So, it must be to have $\gamma \in acQ_1 \times Q \subset H_h$. On the other hand, it is easy to validate $ac\gamma \in P$ and $ac\gamma \in Q_1 \times Q$, or more precisely, $B_2(S_2, ac\gamma) > 0$. Therefore, it is sufficient to show $B_2(S_3, ac\gamma) = 0$.

We use the reduction to absurdity to prove the point above. Suppose $B_2(S_3, ac\gamma) > 0$. Since $\gamma \in H_h$ and $ac\gamma \in P$, we have $ac\gamma \in aH_h$ and $ac\gamma \in Q_2 \times (Q \cup acQ_1)$, which yields $ac\gamma \in Q_2 \times Q$ or $ac\gamma \in Q_2 \times acQ_1$. However, if $ac\gamma \in Q_2 \times Q$ then $\gamma \in acQ_2 \times Q \subset Q \times Q$, or if $ac\gamma \in Q_2 \times acQ_1$ then $\gamma \in Q_2 \times Q_1$. Both cases lead to $B_2(S_2, \gamma) > 0$, contradicting the assumption $\gamma \in P_3$.

From the above two steps, the two inequalities $g_{31} \le g_{21}$ and $g_{32} \le g_{22}$ are proved and hence we get $\bar{g}(S_3) \le \bar{g}(S_2)$. □

Lemma 7 tells us that, when we substitute design S_2 by design S_3, i.e. the elements $a, ab_1, \ldots, ab_{t_0}$ in design S_2, which are out of H_h, are substituted by the elements $c, cb_1, \ldots, cb_{t_0}$, which are in H_h, the $\bar{g}(\cdot)$ value will be reduced. Especially, this procedure can continuously go on till that $t_0 = t$, i.e. $S_3 \subset H_h$, then S_3 has h independent factors and $\bar{g}(S_3) < \bar{g}(S_2)$. If $h > r$, applying Lemma 6 to go the procedure in Lemma 6 but the H_{h-1} in this case has one less independent factor than the previous one. We can repeatedly and alternately go through the procedures of Lemmas 6 and 7 till we construct a design $S_3^* \subset H_r$. Then $\bar{g}(S_3^*) < \bar{g}(S_3)$ and S_3^* has exact r independent factors.

Now, let us return to the proof of Part (a) for the case $r \le q - 1$.

Suppose that S is a design with $2^{r-1} \le s \le 2^r - 1$ factors and $\bar{g}(S)$ is minimized. Obviously, the S has at least r independent factors. If the S has $h\,(> r)$ independent factors, just like the statement in the paragraph after the proof of Lemma 7, we can construct a design S^* such that $\bar{g}(S^*) < \bar{g}(S)$ which contradicts the condition that $\bar{g}(S)$ is minimized. Therefore the S exactly has r independent factors. Noting that $S \subset H_r$ is obvious, the proof of (a) is then completed. □

Proof for Part (b) of Theorem 1

To prove Part (b) of Theorem 1, we need two more lemmas in the following.

Suppose that $S_4 \subset H_q$ is a resolution IV or higher design with s factors, where $2^{r-2} + 1 \le s \le 2^{r-1}$. With a suitable relabelling, we can assume $a \in S_4$. If S_4 has $h + 1$ $(r \le h \le q - 1)$ independent factors, then S_4 has the form

$$S_4 = Q \cup \{a, ab_1, \ldots, ab_t\}, \tag{A.5}$$

where $Q \subset H_h$ and has h independent factors, and $\{b_1, \ldots, b_t\} \subset H_h$. Since S_4 has resolution at least *IV*, aQ and $\{a, ab_1, \ldots, ab_t\}$ are mutually exclusive. Let

$$S_5 = aQ \cup \{a, ab_1, \ldots, ab_t\}. \tag{A.6}$$

Then we have the following result.

LEMMA 8. *Suppose that S_4 and S_5 are defined in (A.5) and (A.6), respectively. Then $\bar{g}(S_5) \leq \bar{g}(S_4)$.*

PROOF. Let $Q_1 = \{a, ab_1, \ldots, ab_t\}$, then $S_4 = Q \cup Q_1$ and $S_5 = aQ \cup Q_1$. From $S_5 \subset \{a, aH_h\}$ and the definition of $\bar{g}(S_5)$, we have

$$\bar{g}(S_5) = \#\{\gamma : \gamma \in H_h, B_2(S_5, \gamma) > 0\}.$$

So, by the definition of $\bar{g}(S_4)$, it suffices to prove that, if $\gamma \in H_h$ and $B_2(S_5, \gamma) > 0$, then $B_2(S_4, \gamma) > 0$ and $\gamma \notin S_4$ or $B_2(S_4, a\gamma) > 0$ and $a\gamma \notin S_4$.

Remind that, if $\gamma \in H_h$ and $B_2(S_5, \gamma) > 0$, then we have $\gamma \in aQ \times aQ$, or $\gamma \in Q_1 \times Q_1$, or $\gamma \in aQ \times Q_1$. Since S_4 has resolution at least *IV*, when $\gamma \in aQ \times aQ\, (= Q \times Q)$ or $\gamma \in Q_1 \times Q_1$, then $B_2(S_4, \gamma) > 0$, which causes $\gamma \notin S_4$, and when $\gamma \in aQ \times Q_1$, then $a\gamma \in Q \times Q_1$ and $B_2(S_4, a\gamma) > 0$, which causes $a\gamma \notin S_4$. □

Lemma 8 tells us that, when we substitute design S_4 by design S_5, i.e. the elements of part Q in design S_4, which is out of F_{qh}, are substituted by the elements aQ, which are in F_{qh}, the $\bar{g}(\cdot)$ value will be reduced.

The following lemma examines the structure of the design that has s factors, resolution *IV* or higher, and r independent factors, where $2^{r-2} + 1 \leq s \leq 2^{r-1}$.

LEMMA 9. *Let $S \subset H_r$ be a design having s factors and resolution IV or higher with $2^{r-2}+1 \leq s \leq 2^{r-1}$, in which there are r independent factors. Then, if $A_i(S) > 0$ for some odd number i, it must have that $A_5(S) > 0$.*

PROOF. Suppose i_0 is the smallest odd number such that $A_{i_0}(S) > 0$. Without loss of generality, we assume $b_1 b_2 \cdots b_{i_0} = I$, where $\{b_1, \ldots, b_{i_0}\} \subset S$ and I is the identity element.

Since S has resolution *IV* or higher, we have $i_0 \geq 5$. We use the reduction to absurdity to prove that surely $i_0 = 5$. Suppose $i_0 \neq 5$, it implies $i_0 \geq 7$, thus we can define the four sets

$$\begin{aligned} Q_1 &= (b_1 b_2 b_3) \times (S \backslash \{b_1, \ldots, b_{i_0}\}), \\ Q_2 &= (b_1 b_4 b_5) \times (S \backslash \{b_1, \ldots, b_{i_0}\}), \\ Q_3 &= (b_2 b_4 b_6) \times (S \backslash \{b_1, \ldots, b_{i_0}\}) \text{ and} \\ Q_4 &= \{b_j b_k,\ 1 \leq j < k \leq i_0\}. \end{aligned}$$

We firstly prove that S, Q_1, Q_2, Q_3 and Q_4 are mutually exclusive. If not, let us suppose that among the five sets there are some two of them the intersection of which is nonempty, say $S \cap Q_1 \neq \emptyset$. Assume $b \in S \cap Q_1$, then there exists some $b' \in S \backslash \{b_1, \ldots, b_{i_0}\}$ such that $b = b_1 b_2 b_3 b'$, which leads that $bb_1 b_2 b_3 b'$ is a defining word of S with length 3 (if $b = b'$ or b_1 or b_2 or b_3) or 5. However, this is impossible under the given assumption for i_0. If there are other two of them whose intersection is nonempty, similarly, we can also find a defining word the length of which is an odd number and smaller than i_0, which is still impossible. By the above arguments, we get

$$\begin{aligned}\#\{S\} + \textstyle\sum_{j=1}^{4} \#\{Q_j\} &= s + 3(s - i_0) + i_0(i_0 - 1)/2 \\ &= 4s + i_0(i_0 - 7)/2 \geq 4s \geq 2^r + 4,\end{aligned}$$

where the third and forth inequalities are from the assumptions $i_0 \geq 7$ and $s \geq 2^{r-2} + 1$, respectively. On the other hand, since $S \subset H_r$, $Q_j \subset H_r$ for $j = 1, 2, 3, 4$, and the five sets are mutually exclusive, we have $\#\{S\} + \sum_{j=1}^{4} \#\{Q_j\} < 2^r$, the contradiction completing the proof of Lemma 9. □

With the preparations above, we come to prove Part (b) of the theorem.

Suppose S is a resolution at least IV design with s factors and $\bar{g}(S)$ is minimized, where $2^{r-2} + 1 \leq s \leq 2^{r-1}$ for some $r \leq q$. Firstly, we prove the first half of Part (b). Since any design $S \subset H_q$ satisfying $2^{q-2} + 1 \leq s \leq 2^{q-1}$ and having resolution at least IV has exactly q independent factors, the first half of Part (b) holds when $r = q$. We only need to consider $r \leq q - 1$.

It is obvious that S has at least r independent factors. If S has $h + 1$ independent factors with $r \leq h \leq q - 1$, we assume that S has the form in (A.5). That is,

$$S = Q \cup \{a, ab_1, \ldots, ab_t\},$$

where Q and $\{a, b_1, \ldots, b_t\}$ satisfy the conditions as in (A.5). Let $Q_1 = \{a, ab_1, \ldots, ab_t\}$ and define $S^* = aQ \cup Q_1$, then $S^* \subset F_{q(h+1)}$. Further let $S^{**} = Q \cup \{b_1, \ldots, b_t\}$, then $S^* = \{a, aS^{**}\}$ and $S^{**} \subset H_h$. It leads that, S^{**} has $s - 1$ factors with $2^{r-2} \leq s - 1 \leq 2^{r-1} - 1$ and among them there are h ones to be independent. Note that, when the range of S is over all the designs with resolution at least IV, then the range of S^{**} is over all the designs with $s - 1$ factors. By the structure of $F_{q(h+1)}$, Lemma 8 and the condition of $\bar{g}(S)$ being minimized, we have

$$\begin{aligned}\bar{g}(S) = \bar{g}(S^*) &= \#\{\gamma : \gamma \in H_q \backslash S^*, B_2(S^*, \gamma) > 0\} \\ &= \#\{\gamma : \gamma \in H_h, B_2(S^*, \gamma) > 0\} \\ &= \#\{\gamma : \gamma \in H_h \backslash S^{**}, B_2(S^*, \gamma) > 0\} + \#\{\gamma : \gamma \in S^{**}, B_2(S^*, \gamma) > 0\} \\ &= \#\{\gamma : \gamma \in H_h \backslash S^{**}, B_2(S^{**}, \gamma) > 0\} + (s - 1) \\ &= \bar{g}(S^{**}) + (s - 1).\end{aligned}$$

Thus, $\bar{g}(S^{**})$ is minimized too. According to Part (a) of the theorem, S^{**} can only have $r-1$ independent factors, contradicting to it having $h\,(\geq r)$ independent factors. This contradiction finishes the proof of the first half of Part (b).

Next, we consider the proof of the second half of Part (b). Now the S has r independent factors. Suppose the S has the form of (A.5) with $h = r-1$, and define S^* as above. Butler (2003) noticed that if $A_i(S) = 0$ for all odd numbers i's, then $S \subset F_{qr}$. Therefore, to finish the proof of the second half, it is sufficient to prove that $A_i(S) = 0$ for all odd numbers i's. If not, according to Lemma 9 and the assumption that S has resolution at least IV, we have $A_5(S) > 0$. In the following we prove that if $A_5(S) > 0$, then $\bar{g}(S^*) < \bar{g}(S)$ which is a contradiction to the assumption that $\bar{g}(S)$ is minimized. By Lemma 8 and its proof, it suffices to show that there exists a $\gamma \in H_{r-1}$ such that $B_2(S^*, \gamma) > 0$, $\gamma \notin S$ with $B_2(S, \gamma) > 0$ and $a\gamma \notin S$ with $B_2(S, a\gamma) > 0$.

Without loss of generality, we assume the factor a appears in the defining word with length 5. By the structure of S, there are two possibilities for this defining word with length 5: one is that, besides a one more factor is from Q_1 and the other three factors are from Q, and the other is that, besides a three more factors are from Q_1 and the other one factor is from Q. After a suitable relabelling, we denote these two possibilities as

$$I = a(ab_1)d_1d_2d_3, \quad \text{where } ab_1 \in Q_1,\ \{d_1, d_2, d_3\} \subset Q$$

and

$$I = a(ab_1)(ab_2)(ab_3)d_1, \quad \text{where } \{ab_1, ab_2, ab_3\} \subset Q_1,\ d_1 \in Q.$$

For the first case, let $\gamma = b_1d_1 = d_2d_3$. It is easy to verify that $B_2(S^*, \gamma) > 0$ and $B_2(S, \gamma) > 0$. Note that $a\gamma = (ab_1)d_1$, where $ab_1 \in Q_1 \subset S$ and $d_1 \in Q \subset S$. Therefore, we have $B_2(S, a\gamma) > 0$. Since the S has resolution IV, $\gamma \notin S$ and $a\gamma \notin S$. For the second case, let $\gamma = (ab_1)(ab_2) = b_3d_1$ and the proof is similar as the first case. Hence the claim that $S \subset F_{qr}$ is proved. Noting that F_{qr} and T_r are isomorphic, then the second half of Part (b) follows. □

Proof for Part (c) of Theorem 1

We first prove that the four designs consisting of the first or last s columns of F_{qr} or T_r are isomorphic. Suppose F'_{qr} consists of the 2^{r-1} columns in F_{qr} in a contrary order. Then we can easily validate

$$F_{qr} = \{q, qH_{r-1}\} \text{ and } F'_{qr} = \{12\cdots(r-1)q, 12\cdots(r-1)qH_{r-1}\}$$

which mean that the design consisting of the first s columns of F_{qr} and the one consisting of the last s columns of F_{qr} are isomorphic. Similarly, the design consisting of the first s columns of T_r and the one consisting of the last s columns of T_r are isomorphic. When F_{qr} and T_r are written in Yates order, from the structures of

F_{qr} and T_r, we have the design consisting of the first s columns of T_r and the one consisting of the first s columns of F_{qr} are isomorphic. Therefore the four designs consisting of the first or last s columns of F_{qr} or T_r are isomorphic.

Suppose that S is a design with s factors and maximizes the sequence (2.4) among all the designs with resolution at least IV and s factors, where $2^{r-2}+1 \le s \le 2^{r-1}$ for some $r \le q$. By the above analysis, clearly, proving Part (c) is equivalent to showing that the unique choice of such S is the design consisting of the first s columns of F_{qr}. In the following we use the mathematical induction to prove this point.

Firstly, we show it holds for $r \le 3$. According to the result of Part (b) just proved, we have $S \subset F_{qr}$. When $s = 1, 2, 3$, under isomorphism, the unique choice of such S is $\{a\}$, $\{a, 1a\}$ and $\{a, 1a, 2a\}$, respectively. Here we remind the mention in Section 2 about resolution at least IV when all the s factors are independent even $s \le 3$. When $s = 4$, according to Part (b) proved above, the number of independent factors in such S is 3 and the choice of S is only $\{a, 1a, 2a, 12a\}$. So, for the four cases of s, such design S is the only one that consists of the first s columns of F_{qr}. Thus the result follows for $r \le 3$.

Next, assume that, for $r \le k$, the fact that the design maximizing (2.4) in all the designs with s factors and resolution at least IV uniquely consists of the first s columns in F_{qr} is true, and come to prove that for $r = k+1$ the fact is true too. By Part (b) of the theorem, we have $S \subset F_{q(k+1)}$. Note that, by Lemma 1 (a) with q being taken as $k+1$ and the condition $2^{k-1}+1 \le s \le 2^k$, for any $\gamma \in H_k$, we have

$$B_2(S, \gamma) = B_2(F_{q(k+1)} \backslash S, \gamma) + s - 2^{k-1} \ge 1$$

and hence $\bar{g}(S) = 2^k - 1$, which is a constant. Therefore, maximizing (2.4) is equivalent to maximizing ${}_2^{\#}C_2(S)$. By Lemma 1 (c) with q being taken as $k+1$, we have that maximizing ${}_2^{\#}C_2(S)$ is equivalent to maximizing the sequence

$$\Big\{ -\bar{g}(F_{q(k+1)} \backslash S),\ {}_2^{\#}C_2(F_{q(k+1)} \backslash S) \Big\}. \tag{A.7}$$

Note that, when $r = k+1$, by the assumptions in Part (c) we have $2^{k-1}+1 \le s \le 2^k$ and the number of factors in $F_{q(k+1)} \backslash S$ is smaller than 2^{k-1}. Applying the inductive assumption for $r \le k$, if $F_{q(k+1)} \backslash S$ consists of the first $2^k - s$ columns in $F_{q(k+1)}$, it uniquely maximizes the sequence (A.7). As we already proved at the beginning of this part, the design consisting of the last $2^k - s$ columns in $F_{q(k+1)}$ columns and the one consisting the first $2^k - s$ columns in $F_{q(k+1)}$ columns are isomorphic. Therefore if we choose $F_{q(k+1)} \backslash S$ to be the one consisting of the last $2^k - s$ columns in $F_{q(k+1)}$, then it also maximizes the sequence (A.7). In this way, the unique choice of such S is the set of the first s columns in $F_{q(k+1)}$, which means that, the result is true for $r = k+1$ and hence it is true for all $r \le q$ by the mathematical induction. This completes the proof of Part (c). □

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